Integrable hierarchy, 3×3 constrained systems, and parametric and peaked stationary solutions

Darryl D. Holm, Zhijun Qiao
T-7 and CNLS, MS B-284, Los Alamos National Laboratory
Los Alamos, NM 87545, USA
E-mails: dholm@lanl.gov qiao@lanl.gov

First version Jan. 28, 2002
Second version July 31, 2002

Abstract

This paper gives a new integrable hierarchy of nonlinear evolution equations. The DP equation: $m_t + um_x + 3mu_x = 0$, $m = u - u_{xx}$, proposed recently by Desgaperis and Procesi [7], is the first one in the negative hierarchy while the first one in the positive hierarchy is: $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. The whole hierarchy is shown Lax-integrable through solving a key matrix equation. To obtain the parametric solutions for the whole hierarchy, we separatedly discuss the negative and the positive hierarchies. For the negative hierarchy, its 3×3 Lax pairs and corresponding adjoint representations are nonlinearized to be Liouville-integrable Hamiltonian canonical systems under the so-called Dirac-Poisson bracket defined on a symplectic submanifold of $\mathbb{R}^{6N}$. Based on the integrability of those finite-dimensional canonical Hamiltonian systems we give the parametric solutions of the all equations in the negative hierarchy. In particular, we obtain the parametric solution of the DP equation. Moreover, for the positive hierarchy, we consider the different constraint and use a similar procudure to the negative case to obtain the parametric solutions of the positive hierarchy. In particular, we give the parametric solution of the 5th-order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. Finally, we discuss the stationary solutions of the 5th-order PDE, and particularly give its four peaked stationary solutions. The stationary solutions may be included in the parametric solution, but the peaked stationary

solutions not. The 5th-order PDE does not have the cusp soliton although it looks like a higher order Harry-Dym equation.

Keywords Hamiltonian system, Matrix equation, Zero curvature representation, Integrable equation, Parametric solution, Peaked stationary solution.

AMS Subject: 35Q53; 58F07; 35Q35
PACS: 03.40.Gc; 03.40Kf; 47.10.+g

1 Introduction

The inverse scattering transformation (IST) method is a powerful tool to solve integrable nonlinear evolution equations (NLEEs) [8]. This method has been successfully applied to give soliton solutions of the integrable NLEEs. These examples include the well-known KdV equation [16], which is related to a 2nd order operator (i.e. Hill operator) spectral problem [17, 19], the remarkable AKNS equations [1, 2], which is associated with the Zakharov-Shabat (ZS) spectral problem [23], and other higher dimensional integrable equations.

To look for as many integrable systems as possible has been an important topic In the theory of integrable system. Kaup [13] studied the inverse scattering problem for cubic eigenvalue equations of the form $\psi_{xxx} + 6Q\psi_x + 6R\psi = \lambda\psi$, and showed a 5th order partial differential equation (PDE) $Q_t + Q_{xxxxx} + 30(Q_{xxx}Q + \frac{5}{2}Q_{xx}Q_x) + 180Q_xQ^2 = 0$ (called the KK equation) integrable. Afterwards, Kuperschmidt [15] constructed a super-KdV equation and presented the integrability of the equation through giving bi-Hamiltonian property and Lax form. In 1984 Konopelchenko and Dubrovsky [14] presented a 5th order equation $u_t = (u^{-2/3})_{xxxxx}$ and pointed out that this equation is a reduction of some $2+1$ dimensional equation. We have found the parametric solution and some traveling wave solutions of this equation [11].

Recently, Degasperis and Procesi [7] proposed a new integrable equation: $m_t + um_x + 3mu_x = 0$, $m = u - u_{xx}$, called the DP equation, which has the peaked soliton solution. The DP equation is an extention of the Camassa-Holm (CH) equation [4], and is proven to be associated with a 3rd order spectral problem [6]: $\psi_{xxx} = \psi_x - \lambda m\psi$ and to have some relationship to a canonical Hamiltonian system under a new nonlinear Poisson bracket (called Peakon Bracket) [9].

The purpose of the present paper has two folds:

- extend the DP equation to a new integrable hierarchy of NLEEs through studying the functional gradient of the spectral problem and a pair of Lenard's operators;

- connect the DP equation to some finite-dimensional integrable system, give its parametric solution from the view point of constraint, and furthermore study the parametric solution, stationary solution and cusp soliton solution.

The whole paper is organized as follows. Next section is saying how to connect a spectral problem to the DP equation and how to cast it into a new hierarchy of NLEEs, and is also giving the pair of Lenards operators for the whole hierarchy. In section 3, we construct the zero curvature representations for this new hierarchy through solving a key 3×3 matrix equation, and therefore this hierarchy is integrable. In particular, we give the matrix Lax pair of the DP equation, which is equivalent to the form in Ref. [6], as well as the Lax pair for a 5th-order equation $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. We will see that the DP equation is included in the negative hierarchy while the 5th-order equation in the positive hierarchy. To obtain the parametric solutions for the whole hierarchy, we separatedly discuss the negative and the positive hierarchies. In section 4, we deal with the negative hierarchy. Its 3×3 Lax pairs and corresponding adjoint representations are nonlinearized to be Liouville-integrable Hamiltonian canonical systems under the so-called Dirac-Poisson bracket defined on a symplectic submanifold of $\mathbb{R}^{6N}$. Based on the integrability of those finite-dimensional canonical Hamiltonian systems we give the parametric solutions of the all equations in the negative hierarchy. In particular, we obtain the parametric solution of the DP equation. Section 5 copes with the positive hierarchy. We consider the different constraint between the potential and the eigenfunctions. Under this constraint the 3×3 Lax pairs and corresponding adjoint representations of the positive hierarchy are nonlinearized to be Liouville-integrable Hamiltonian canonical systems in the whole $\mathbb{R}^{6N}$. Then we obtain the parametric solutions of the positive hierarchy. In particular, we give the parametric solution of the 5th-order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. Finally, in section 6 we discuss the stationary solutions of the 5th-order PDE, and particularly give its four peaked stationary solutions. The stationary solutions may be included in the parametric solution, but the peaked stationary solutions not. The 5th-order PDE does not have the cusp soliton, either.

2 Spectral problems and Lenards operators

Let us consider the following spectral problem

$$\psi_{xxx} = \frac{1}{\alpha^2}\psi_x - \lambda m \psi \tag{1}$$

and its adjoint problem

$$\psi^*_{xxx} = \frac{1}{\alpha^2}\psi^*_x + \lambda m \psi^*, \tag{2}$$

where λ is a spectral parameter, m is a scalar potential function, ψ and ψ^* are the spectral wave functions corresponding to the same λ, ψ^* is not conjugate of ψ, and $\alpha = constant$. Then, we have

$$\nabla\lambda = \frac{\delta\lambda}{\delta m} = \frac{\lambda\psi\psi^*}{E} \tag{3}$$

where

$$E = -\int_{-\infty}^{\infty} m\psi\psi^* dx = const.$$

Here during our computation about the functional gradient $\frac{\delta\lambda}{\delta m}$ of the spectral parameter λ with respect to the potential m, we need the boundary conditions of decaying at infinities or periodicity condition for the potential function m. A general calculated method can be seen in Refs. [5, 21].

Now, we denote the function $\nabla_1\lambda$ by

$$\nabla_1\lambda = \frac{\lambda(\psi\psi^*_x - \psi^*\psi_x)}{E}, \tag{4}$$

Then, we have the following equality:

$$\bar{K}(\nabla\lambda, \nabla_1\lambda)^T = \lambda\bar{J}(\nabla\lambda, \nabla_1\lambda)^T$$

where $\bar{K}$ and $\bar{J}$ are two matrix operators

$$\begin{aligned}
\bar{K} &= \begin{pmatrix} -4\partial + 5\alpha^2\partial^3 - \alpha^4\partial^5 & 0 \\ 0 & \partial^3 - \frac{1}{\alpha^2}\partial \end{pmatrix}, \\
\bar{J} &= \begin{pmatrix} 0 & 3\alpha^4(2m\partial + \partial m) \\ m\partial + 2\partial m & 0 \end{pmatrix}, \\
\partial &= \frac{\partial}{\partial x},
\end{aligned}$$

or we rewrite Eq. (5) as the following Lenard spectral problem form

$$K\nabla\lambda = \lambda^2 J\nabla\lambda, \tag{5}$$

where

$$\begin{aligned}
K &= 4\partial - 5\alpha^2\partial^3 + \alpha^4\partial^5, \\
J &= 3\alpha^6(2m\partial + \partial m)(-\alpha^2\partial^3 + \partial)^{-1}(m\partial + 2\partial m).
\end{aligned}$$

Without loss of generality, we assume $\alpha = 1$ below. Then, K, J read

$$K = 4\partial - 5\partial^3 + \partial^5, \tag{6}$$

$$J = 3(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m). \tag{7}$$

Hint: Here we do not care about the Hamiltonian properties of the operators K, J, but need

$$\begin{aligned} K^{-1} &= (\partial - \partial^3)^{-1}(4 - \partial^2)^{-1}, \\ J^{-1} &= \frac{1}{27} m^{-2/3}\partial^{-1}m^{-1/3}(\partial - \partial^3)m^{-1/3}\partial^{-1}m^{-2/3}. \end{aligned}$$

They yield

$$\begin{aligned} &\mathcal{L} = J^{-1}K = \frac{1}{27} m^{-2/3}\partial^{-1}m^{-1/3}(\partial - \partial^3)m^{-1/3}\partial^{-1}m^{-2/3}(4\partial - 5\partial^3 + \partial^5), \\ &\mathcal{L}^{-1} = K^{-1}J = 3(\partial - \partial^3)^{-1}(4 - \partial^2)^{-1}(2m\partial + \partial m)(\partial - \partial^3)^{-1}(m\partial + 2\partial m), \end{aligned}$$

which are actually the two recursive operators we need in the next section.

3 Zero curvature representations and an integrable hierarchy

Letting $\psi = \psi_1$, we change Eq. (1) to a 3×3 matrix spectral problem

$$\Psi_x = U(m, \lambda)\Psi, \tag{8}$$

$$U(m, \lambda) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -m\lambda & 1 & 0 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}. \tag{9}$$

Apparently, the Gateaux derivative matrix $U_*(\xi)$ of the spectral matrix U in the direction $\xi \in C^\infty(\mathbb{R})$ at point m is

$$U_*(\xi) \triangleq \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} U(m + \epsilon\xi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\xi\lambda & 0 & 0 \end{pmatrix} \tag{10}$$

which is obviously an injective homomorphism.

For any given C^∞-function G, we construct the following 3×3 matrix equation with respect to $V = V(G)$

$$V_x - [U, V] = U_*(KG - \lambda^2 JG). \tag{11}$$

Theorem 1 *For the spectral problem (8) and an arbitrary C^∞-function G, the matrix equation (11) has the following solution*

$$V=\lambda\left(\begin{array}{ccc} \Gamma G+3\lambda\partial\Theta^{-1}\Upsilon G & 3G_x-3\lambda\Theta^{-1}\Upsilon G & -6G \\ \Gamma G_x+3\lambda(\partial^2\Theta^{-1}\Upsilon G+2mG) & -2(G-G_{xx}) & -3G_x-3\lambda\Theta^{-1}\Upsilon G \\ \Gamma G_{xx}+3\lambda(\partial+\lambda m)\Theta^{-1}\Upsilon G & -\Theta G-3\lambda(\partial^{-1}\Upsilon G-2mG) & -2G-G_{xx}-3\lambda\partial\Theta^{-1}\Upsilon G \end{array}\right) \tag{12}$$

where $\Theta=\partial-\partial^3,\ \Upsilon=m\partial+2\partial m,\ \Gamma=4-\partial^2$. *Therefore,* $J=3\Upsilon^*\Theta^{-1}\Upsilon,\ K=\Gamma\Theta$.

Proof: Let

$$V=\left(\begin{array}{ccc} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{array}\right),$$

and substitute it into Eq. (11). That is a over-determined equation. Using some calculation technique [20], we obtain the following results:

$$\begin{aligned}
V_{11} &= \lambda\Gamma G+3\lambda^2\partial\Theta^{-1}\Upsilon G, \\
V_{12} &= 3\lambda G_x-3\lambda^2\Theta^{-1}\Upsilon G, \\
V_{13} &= -6\lambda G, \\
V_{21} &= \lambda\Gamma G_x+3\lambda^2(\partial^2\Theta^{-1}\Upsilon G+2mG), \\
V_{22} &= -2\lambda(G-G_{xx}), \\
V_{23} &= -3\lambda G_x-3\lambda^2\Theta^{-1}\Upsilon G, \\
V_{31} &= \lambda\Gamma G_{xx}+3\lambda^2(\partial+\lambda m)\Theta^{-1}\Upsilon G, \\
V_{32} &= -\lambda\Theta G-3\lambda^2(\partial^{-1}\Upsilon G-2mG), \\
V_{33} &= -2\lambda G-\lambda G_{xx}-3\lambda^2\partial\Theta^{-1}\Upsilon G,
\end{aligned}$$

which completes the proof.

Theorem 2 *Let* $G_0\in Ker\ J=\{G\in C^\infty(\mathbb{R})\mid JG=0\}$ *and* $G_{-1}\in Ker\ K=\{G\in C^\infty(\mathbb{R})\mid KG=0\}$. *We define the Lenard's sequences as follows*

$$G_j=\mathcal{L}^jG_0=\mathcal{L}^{j+1}G_{-1},\ j\in\mathbb{Z}. \tag{13}$$

Then,

1. *the all vector fields* $X_k=JG_k,\ k\in\mathbb{Z}$ *satisfy the following commutator representation*

$$V_{k,x}-[U,V_k]=U_*(X_k),\ \forall k\in\mathbb{Z}; \tag{14}$$

2. *the following hierarchy of nonlinear evolution equations*

$$m_{t_k} = X_k = JG_k, \ \forall k \in \mathbb{Z}, \tag{15}$$

possesses the zero curvature representation

$$U_{t_k} - V_{k,x} + [U, V_k] = 0, \ \forall k \in \mathbb{Z}, \tag{16}$$

where

$$V_k = \sum V(G_j)\lambda^{2(k-j-1)}, \quad \sum = \begin{cases} \sum_{j=0}^{k-1}, & k > 0, \\ 0, & k = 0, \\ -\sum_{j=k}^{-1}, & k < 0, \end{cases} \tag{17}$$

and $V(G_j)$ is given by Eq. (12) with $G = G_j$.

Proof:

1. For $k = 0$, it is obvious. For $k < 0$, we have

$$\begin{aligned} V_{k,x} - [U, V_k] &= -\sum_{j=k}^{-1} \left(V_x(G_j) - [U, V(G_j)]\right)\lambda^{2(k-j-1)} \\ &= -\sum_{j=k}^{-1} U_* \left(KG_j - \lambda^2 KG_{j-1}\right)\lambda^{2(k-j-1)} \\ &= U_* \left(\sum_{j=k}^{-1} KG_{j-1}\lambda^{2(k-j)} - KG_j\lambda^{2(k-j-1)}\right) \\ &= U_* \left(KG_{k-1} - KG_{-1}\lambda^{2k}\right) \\ &= U_*(KG_{k-1}) \\ &= U_*(X_k). \end{aligned}$$

   For the case of $k > 0$, it is similar to prove.

2. Noticing $U_{t_k} = U_*(m_{t_k})$, we obtain

$$U_{t_k} - V_{k,x} + [U, V_k] = U_*(m_{t_k} - X_k).$$

   The injectiveness of U_* implies item 2 holds.

So, the hierarchy (15) has Lax pair and is therefore integrable. In particular, through choosing $G_{-1} = -\frac{1}{6} \in Ker\ K$, (15) reads

$$m_{t_k} = -J\mathcal{L}^{k+1} \cdot \frac{1}{6}, \quad k = -1, -2, ..., \tag{18}$$

where $\mathcal{L} = J^{-1}K$. Set $m = u - u_{xx}$, then it is easy to see the first equation in the hierarchy is exactly the DP equation [7]

$$m_t + um_x + 3mu_x = 0,\ t = t_{-1}. \tag{19}$$

This equation has the following Lax pair:

$$\begin{aligned} \Psi_x &= U(m,\lambda)\Psi, \\ \Psi_t &= V(m,\lambda)\Psi, \end{aligned} \tag{20}$$

where $U(m,\lambda)$ is defined by Eq. (9), and $V(m,\lambda)$ is given by

$$V(m,\lambda) = \begin{pmatrix} u_x + \frac{2}{3}\lambda^{-1} & -u & -\lambda^{-1} \\ u & -\frac{1}{3}\lambda^{-1} & -u \\ u_x + um\lambda & 0 & -u_x - \frac{1}{3}\lambda^{-1} \end{pmatrix}, \tag{21}$$

which can be changed to the form in Ref. [6]

$$\psi_t + \lambda^{-1}\psi_{xx} + u\psi_x - \left(u_x + \frac{2}{3}\lambda^{-1}\right)\psi = 0,\ \psi = \psi_1. \tag{22}$$

Let us choose a kernel element G_0 from Ker$J$: $G_0 = m^{-\frac{2}{3}}$. Then Eq. (15) reads the following integrable hierarchy

$$m_{t_k} = J\mathcal{L}^k m^{-\frac{2}{3}},\ k = 0, 1, 2, ... \tag{23}$$

In particular, the equation

$$m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}, \tag{24}$$

has the Lax pair:

$$\begin{aligned} \Psi_x &= U(m,\lambda)\Psi, \\ \Psi_t &= V_1(m,\lambda)\Psi, \end{aligned} \tag{25}$$

where $U(m,\lambda)$ is defined by Eq. (9), and $V_1(m,\lambda)$ is given by

$$V_1(m,\lambda) = \lambda \begin{pmatrix} \Gamma m^{-\frac{2}{3}} & 3(m^{-\frac{2}{3}})_x & -6m^{-\frac{2}{3}} \\ \Gamma(m^{-\frac{2}{3}})_x + 6\lambda m^{\frac{1}{3}} & 2(m^{-\frac{2}{3}})_{xx} - 2m^{-\frac{2}{3}} & -3(m^{-\frac{2}{3}})_x \\ \Gamma(m^{-\frac{2}{3}})_{xx} & (m^{-\frac{2}{3}})_{xxx} - (m^{-\frac{2}{3}})_x + 6\lambda m^{\frac{1}{3}} & -2m^{-\frac{2}{3}} - (m^{-\frac{2}{3}})_{xx} \end{pmatrix} \tag{26}$$

with the operator $\Gamma = 4 - \partial^2$. Eq. (24) is therefore a new integrable equation.

In the next two sections we will give parametric solutions for the negative order hierarchy (18) and the positive order hierarchy (23).

4 Parametric solution of the negative order hierarchy (18)

To get the parametric solution, we use the constrained method which leads finite dimensional integrable systems to the PDEs. Becasue Eq. (8) is a 3rd order eigenvalue problem, we have to investigate itself together with its adjoint problem when we adopt the nonlinearized procedure [5]. Ma and Strampp [18] ever studied the AKNS and its its adjoint problem, a 2×2 case, by using the so-called symmetry constraint method. Now, we are discussing a 3×3 problem related to the hierarchy (15). Let us return to the spectral problem (8) and consider its adjoint problem

$$\Psi^*_x = \begin{pmatrix} 0 & 0 & m\lambda \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \Psi^*, \ \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}, \tag{27}$$

where $\psi^* = \psi_3^*$.

4.1 Nonlinearized spectral problems on a symplectic submanifold

Let λ_j $(j = 1, ..., N)$ be N distinct spectral values of (8) and (27), and q_{1j}, q_{2j}, q_{3j} and p_{1j}, p_{2j}, p_{3j} be the corresponding spectral functions, respectively. Then we have

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -m\Lambda q_1 + q_2, \end{aligned} \tag{28}$$

and

$$\begin{aligned} p_{1x} &= m\Lambda p_3, \\ p_{2x} &= -p_1 - p_3, \\ p_{3x} &= -p_2, \end{aligned} \tag{29}$$

where $\Lambda = diag(\lambda_1, ..., \lambda_N), \ q_k = (q_{k1}, q_{k2}, ..., q_{kN})^T, \ p_k = (p_{k1}, p_{k2}, ..., p_{kN})^T, \ k = 1, 2, 3$.

Now, we consider the following $(6N-2)$-dimensional symplectic submanifold in $\mathbb{R}^{6N}$:

$$M = \{(p, q)^T \in \mathbb{R}^{6N} | \ F = 0, \ G = 0\} \tag{30}$$

where $p = (p_1, p_2, p_3)^T, \ q = (q_1, q_2, q_3)^T$, $F = \langle \Lambda q_1, p_3 \rangle - 1$, $G = \langle \Lambda q_2, p_3 \rangle - \langle \Lambda q_3, p_2 \rangle + \langle \Lambda q_2, p_1 \rangle$, and $\langle \cdot, \cdot \rangle$ stands for the standard inner product in $\mathbb{R}^N$.

When we restrict the above two systems in $\mathbb{R}^{6N}$ to the submanifold M, we obtain a constraint of m relationship to the spectral function p, q:

$$m = 2\frac{\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle}{\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle}. \tag{31}$$

Notice: $\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle \neq 0$ is necessary because it assures M is a symplectic submanifold in $\mathbb{R}^{6N}$.

Under the constraint (31) the two systems (28)and (29) are nonlinearized as follows:

$$\begin{aligned} q_{1x} &= q_2, \\ q_{2x} &= q_3, \\ q_{3x} &= -2\tfrac{\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_3\rangle - \langle \Lambda q_3, p_1\rangle}{\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle}\Lambda q_1 + q_2; \end{aligned} \tag{32}$$

and

$$\begin{aligned} p_{1x} &= 2\tfrac{\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_3\rangle - \langle \Lambda q_3, p_1\rangle}{\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle}\Lambda p_3, \\ p_{2x} &= -p_1 - p_3, \\ p_{3x} &= -p_2. \end{aligned} \tag{33}$$

They are forming a $(6N-2)$ dimensional nonlinear system on M with respect to p, q. Is it integrable? To see this, in $\mathbb{R}^{6N}$ we modify the usual standard Poisson bracket [3] of two functions F_1, F_2 as follows:

$$\{F_1, F_2\} = \sum_{i=1}^{3}\left(\left\langle \frac{\partial F_1}{\partial q_i}, \frac{\partial F_2}{\partial p_i}\right\rangle - \left\langle \frac{\partial F_1}{\partial p_i}, \frac{\partial F_2}{\partial q_i}\right\rangle\right) \tag{34}$$

which is still antisymmetric, bilinear and satisfies the Jacobi identity.

Obviously,

$$\{F, G\} = \left\langle \Lambda^2 q_2, p_3\right\rangle + \left\langle \Lambda^2 q_1, p_2\right\rangle \neq 0. \tag{35}$$

Because we are discussing the system on the submanifold M, we need to introduce the so-called Dirac-Poisson bracket of two functions f, g on M:

$$\{f, g\}_D = \{f, g\} + \frac{1}{\{F, G\}}\left(\{f, F\}\{G, g\} - \{f, G\}\{F, g\}\right) \tag{36}$$

which is satisfying the Jacobi identity.

Now, we choose a simple Hamiltonian

$$H = \langle q_2, p_1 + p_3\rangle + \langle q_3, p_2\rangle, \tag{37}$$

then, the two systems (32) and (33) have the canonical Hamiltonian form on M:

$$\begin{aligned} q_{1j,x} &= \{q_{1j}, H\}_D, \\ q_{2j,x} &= \{q_{2j}, H\}_D, \\ q_{3j,x} &= \{q_{3j}, H\}_D; \\ p_{1j,x} &= \{p_{1j}, H\}_D, \\ p_{2j,x} &= \{p_{2j}, H\}_D, \\ p_{3j,x} &= \{p_{3j}, H\}_D, \end{aligned} \tag{38}$$

which can be in a brief form rewritten as:

$$\begin{aligned} q_x &= \{q, H\}_D, \\ p_x &= \{p, H\}_D. \end{aligned} \tag{39}$$

In this calculation procedure, we have used

$$\begin{aligned} \{H, G\} &= 2\Big(\langle \Lambda q_2, p_2 \rangle - \langle \Lambda q_3, p_1 + p_3 \rangle \Big), \\ \{H, F\} &= 0. \end{aligned}$$

It is easy to check that $H_x = 0$, i.e. H is invariant along the flow (39). Assume $H = C_D$ (C_D is constant) along this flow and

$$u = \frac{1}{2} \left(\langle q_1, p_2 \rangle + \langle q_2, p_3 \rangle \right) - \frac{1}{2} H, \tag{40}$$

then we have

$$u - u_{xx} = m, \tag{41}$$

which is exactly related to the DP equation (19).

To show the integrability of canonical system (39), we need to consider the nonlinearization of the time part of the Lax representations.

4.2 Nonlinearized time part on this submanifold

Let us turn to the time part (20) of the Lax pair for the TD equation (19). Then the corresponding adjoint problem reads

$$\Psi_t^* = \begin{pmatrix} -u_x - \frac{2}{3}\lambda^{-1} & -u & -u_x - um\lambda \\ u & \frac{1}{3}\lambda^{-1} & 0 \\ \lambda^{-1} & u & u_x + \frac{1}{3}\lambda^{-1} \end{pmatrix} \Psi^*, \;\; \Psi^* = \begin{pmatrix} \psi_1^* \\ \psi_2^* \\ \psi_3^* \end{pmatrix}. \tag{42}$$

We also consider the constrained system of the time part on M. Thus, under the constraints (31), (40), and

$$u_x = \frac{1}{2}\left(\langle q_3 - q_1, p_3\rangle - \langle q_1, p_1\rangle\right), \tag{43}$$

Eqs. (20) and (42) are nonlinearized as:

$$\begin{aligned} q_{1t} &= u_x q_1 + \tfrac{2}{3}\Lambda^{-1} q_1 - u q_2 - \Lambda^{-1} q_3, \\ q_{2t} &= u q_1 - \tfrac{1}{3}\Lambda^{-1} q_2 - u q_3, \\ q_{3t} &= u_x q_1 + u m \Lambda q_1 - \tfrac{1}{3}\Lambda^{-1} q_3 - u_x q_3; \end{aligned} \tag{44}$$

and

$$\begin{aligned} p_{1t} &= -u_x p_1 - \tfrac{2}{3}\Lambda^{-1} p_1 - u p_2 - u_x p_3 - u m \Lambda p_3, \\ p_{2t} &= u p_1 + \tfrac{1}{3}\Lambda^{-1} p_2, \\ p_{3t} &= \Lambda^{-1} p_1 + u p_2 + u_x p_3 + \tfrac{1}{3}\Lambda^{-1} p_3, \end{aligned} \tag{45}$$

respectively, where each q_k, p_k and Λ are the same as section 2, and Λ^{-1} is the inverse of Λ.

Let

$$\begin{aligned} I = {} & \frac{2}{3}\left\langle \Lambda^{-1} q_1, p_1\right\rangle - \frac{1}{3}\left\langle \Lambda^{-1} q_2, p_2\right\rangle - \frac{1}{3}\left\langle \Lambda^{-1} q_3, p_3\right\rangle - \left\langle \Lambda^{-1} q_3, p_1\right\rangle \\ & + \frac{1}{4}\langle q_1, p_2\rangle^2 - \frac{1}{4}\langle q_2, p_3\rangle^2 - \frac{1}{2} H \langle q_1, p_2\rangle \\ & - u\left(\langle q_2, p_1\rangle + \langle q_3, p_2\rangle - H\right) - u_x^2, \end{aligned} \tag{46}$$

then the two systems (44) and (45) are expressed in a canonical Hamiltonian form on M:

$$\begin{aligned} q_t &= \{q, I\}_D, \\ p_t &= \{p, I\}_D, \end{aligned} \tag{47}$$

where $p = (p_1, p_2, p_3)^T, q = (q_1, q_2, q_3)^T$.

In the above calculations, we use the following equalities:

$$\begin{aligned} F &= \langle \Lambda q_1, p_3\rangle - 1 = 0, \\ G &= \langle \Lambda q_2, p_1 + p_3\rangle - \langle \Lambda q_3, p_2\rangle = 0, \\ F_x &= \langle \Lambda q_2, p_3\rangle - \langle \Lambda q_1, p_2\rangle = 0, \\ F_{xx} &= \langle \Lambda q_3, p_3\rangle - 2\langle \Lambda q_2, p_2\rangle + \langle \Lambda q_1, p_1\rangle + 1 = 0, \\ \{G, I\} &= 2u\Big(\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle\Big), \\ \{F, I\} &= 0. \end{aligned}$$

Theorem 3

$$\{H, I\}_D = 0, \tag{48}$$

that is, two Hamiltonian flows commute on M.

Proof: By the definition,

$$\{H, I\}_D = \{H, I\} + \frac{1}{\{F, G\}} \left(\{H, F\}\{G, I\} - \{H, G\}\{F, I\} \right), \tag{49}$$

we need to compute each Poisson bracket in this equality.

$$\begin{aligned}
\{H, I\} &= u_x \left(\langle p_2, q_1 - q_3 \rangle - \langle q_2, p_1 \rangle \right) + u \left(\langle p_3, q_1 - q_3 \rangle - \langle q_1, p_1 \rangle \right) = 0, \\
\{H, F\} &= \langle \Lambda q_1, p_2 \rangle - \langle \Lambda q_2, p_3 \rangle = (- \langle \Lambda q_1, p_3 \rangle + 1)_x = 0, \\
\{F, I\} &= 2u_x(\langle \Lambda q_1, p_3 \rangle - 1) + u(- \langle \Lambda q_1, p_3 \rangle + 1)_x = 0,
\end{aligned}$$

complete the proof.

By Theorem 2, we know that the hierarchy (18) has the Lax representation:

$$\Psi_x = U(m, \lambda)\Psi, \tag{50}$$

$$\Psi_{t_j} = -\sum_{k=j}^{-1} V(G_k)\lambda^{2(j-k-1)}\Psi, \ j < 0, \tag{51}$$

where $V(G_k)$ is given by Eq. (12) with $G = G_k$.

In last section, we have investigated the nonlinearized systems of spectral problem and its adjoint. Now, we study the nonlinearizations of time part (51) and its adjoint problem:

$$\Psi^*_{t_j} = \sum_{k=j}^{-1} V^T(G_k)\lambda^{2(j-k-1)}\Psi^*, \ j < 0, \tag{52}$$

where $V^T(G_k)$ is the transpose of $V(G_k)$.

Let $\Psi_k = (q_{1k}, q_{2k}, q_{3k})^T$, $\Psi^*_k = (p_{1k}, p_{2k}, p_{3k})^T$ be the eigenfunctions corresponding to N the eigenvalues λ_k ($k = 1, ..., N$) of (8) and (27). Let us start from the constraint $G_{-1} = -\frac{1}{6}\sum_{j=1}^{N} \nabla\lambda_j$. This constraint is giving the symplectic submanifold M we need. Let the two antisymmetric operators act on this constraint, we have:

$$G_j = -\frac{1}{6} \left\langle \Lambda^{2j+3} q_1, p_3 \right\rangle, \ j < 0. \tag{53}$$

Therefore, a complicated calculation yields the following formulations:

$$
\begin{aligned}
G_j - G_{j,xx} &= \frac{1}{6}\Big(\left\langle \Lambda^{2j+3} q_1, p_1 \right\rangle + \left\langle \Lambda^{2j+3} q_3, p_3 \right\rangle - 2 \left\langle \Lambda^{2j+3} q_2, p_2 \right\rangle \Big), \\
\Gamma G_j &= \frac{1}{6}\Big(\left\langle \Lambda^{2j+3} q_1, p_1 \right\rangle + \left\langle \Lambda^{2j+3} q_3, p_3 \right\rangle - 2 \left\langle \Lambda^{2j+3} q_2, p_2 \right\rangle - 3 \left\langle \Lambda^{2j+3} q_1, p_3 \right\rangle \Big), \\
\Gamma G_{j,x} &= \frac{1}{2}\Big(\left\langle \Lambda^{2j+3} q_1, p_2 \right\rangle + \left\langle \Lambda^{2j+3} q_2, p_1 \right\rangle - \left\langle \Lambda^{2j+3} q_3, p_2 \right\rangle \Big), \\
\Gamma G_{j,xx} &= \frac{1}{2}\Big[m\Big(\left\langle \Lambda^{2j+4} q_1, p_2 \right\rangle + \left\langle \Lambda^{2j+4} q_2, p_3 \right\rangle \Big) \\
&\quad + 2 \left\langle \Lambda^{2j+3} q_3, p_1 \right\rangle + \left\langle \Lambda^{2j+3} q_3, p_3 \right\rangle - \left\langle \Lambda^{2j+3} q_1, p_1 + p_3 \right\rangle \Big], \\
\Theta G_j &= \frac{1}{2}\Big(\left\langle \Lambda^{2j+3} q_2, p_1 + p_3 \right\rangle - \left\langle \Lambda^{2j+3} q_3, p_2 \right\rangle \Big), \\
\partial^{-1} \Upsilon G_j - 2m G_j &= -\frac{1}{6}\Big(\left\langle \Lambda^{2j+2} q_2, p_1 + p_3 \right\rangle + \left\langle \Lambda^{2j+2} q_3, p_2 \right\rangle \Big), \\
\partial^2 \Theta^{-1} \Upsilon G_j + 2m G_j &= \frac{1}{6}\Big(\left\langle \Lambda^{2j+2} q_2, p_1 \right\rangle + \left\langle \Lambda^{2j+2} q_3, p_2 \right\rangle - \left\langle \Lambda^{2j+2} q_1, p_2 \right\rangle \Big), \\
\Theta^{-1} \Upsilon G_j &= -\frac{1}{6}\Big(\left\langle \Lambda^{2j+2} q_1, p_2 \right\rangle + \left\langle \Lambda^{2j+2} q_2, p_3 \right\rangle \Big).
\end{aligned}
$$

Substituting these equalities to Eqs. (51) and (52), with a similar computational method to section 3 we find the nonlinearized systems of the time part (51) and the adjoint time part (52) are cast in a canonical Hamiltonian system on the $(6N-2)$-dimensional symplectic submanifold M:

$$
\begin{aligned}
q_{t_k} &= \{q, F_k\}_D, \\
p_{t_k} &= \{p, F_k\}_D,
\end{aligned}
\qquad k = -1, -2, -3, ..., \tag{54}
$$

where

$$
\begin{aligned}
F_k =\; & \frac{2}{3} \left\langle \Lambda^{2k+1} q_1, p_1 \right\rangle - \frac{1}{3} \left\langle \Lambda^{2k+1} q_2, p_2 \right\rangle - \frac{1}{3} \left\langle \Lambda^{2k+1} q_3, p_3 \right\rangle - \left\langle \Lambda^{2k+1} q_3, p_1 \right\rangle \\
& + \sum_{j=k}^{-2} \Big[-\frac{1}{12} \langle \Lambda^{2j+3} q_1, p_1 \rangle \left\langle \Lambda^{2(k-j)-1} q_1, p_1 \right\rangle - \frac{1}{3} \langle \Lambda^{2j+3} q_2, p_2 \rangle \left\langle \Lambda^{2(k-j)-1} q_2, p_2 \right\rangle - \frac{1}{4} \langle \Lambda^{2j+3} q_3, p_3 \rangle \left\langle \Lambda^{2(k-j)-1} q_3, p_3 \right\rangle \\
& + \frac{1}{3} \langle \Lambda^{2j+3} q_2, p_2 \rangle \left\langle \Lambda^{2(k-j)-1} q_1, p_1 \right\rangle - \frac{1}{6} \langle \Lambda^{2j+3} q_3, p_3 \rangle \left\langle \Lambda^{2(k-j)-1} q_1, p_1 \right\rangle + \frac{1}{3} \langle \Lambda^{2j+3} q_3, p_3 \rangle \left\langle \Lambda^{2(k-j)-1} q_2, p_2 \right\rangle \\
& - \frac{1}{4} \langle \Lambda^{2j+3} q_1, p_2 \rangle \left\langle \Lambda^{2(k-j)-1} q_1, p_2 \right\rangle + \frac{1}{4} \langle \Lambda^{2j+3} q_1, p_3 \rangle \left\langle \Lambda^{2(k-j)-1} q_1, p_3 \right\rangle + \frac{1}{4} \langle \Lambda^{2j+3} q_2, p_3 \rangle \left\langle \Lambda^{2(k-j)-1} q_2, p_3 \right\rangle \\
& + \frac{1}{2}\Big(\langle \Lambda^{2j+3} q_1, p_2 \rangle - \langle \Lambda^{2j+3} q_2, p_3 \rangle \Big)\Big(\left\langle \Lambda^{2(k-j)-1} q_3, p_2 \right\rangle - \left\langle \Lambda^{2(k-j)-1} q_2, p_1 \right\rangle \Big) \\
& + \frac{1}{2} \langle \Lambda^{2j+3} q_1, p_3 \rangle \Big(\left\langle \Lambda^{2(k-j)-1} q_1, p_1 \right\rangle - 2 \left\langle \Lambda^{2(k-j)-1} q_3, p_1 \right\rangle - \left\langle \Lambda^{2(k-j)-1} q_3, p_3 \right\rangle \Big) \Big] \\
& + \sum_{j=k}^{-1} \Big[-\frac{1}{4} \langle \Lambda^{2j+2} q_1, p_1 \rangle \left\langle \Lambda^{2(k-j)} q_1, p_1 \right\rangle - \frac{1}{4} \langle \Lambda^{2j+2} q_3, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_3, p_3 \right\rangle \\
& + \frac{1}{4} \langle \Lambda^{2j+2} q_1, p_2 \rangle \left\langle \Lambda^{2(k-j)} q_1, p_2 \right\rangle - \frac{1}{4} \langle \Lambda^{2j+2} q_1, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_1, p_3 \right\rangle - \frac{1}{4} \langle \Lambda^{2j+2} q_2, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_2, p_3 \right\rangle \\
& - \frac{1}{2}\Big(\langle \Lambda^{2j+2} q_1, p_2 \rangle + \langle \Lambda^{2j+2} q_2, p_3 \rangle \Big)\Big(\left\langle \Lambda^{2(k-j)} q_3, p_2 \right\rangle + \left\langle \Lambda^{2(k-j)} q_2, p_1 \right\rangle \Big) \\
& + \frac{1}{2} \langle \Lambda^{2j+2} q_1, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_3, p_3 \right\rangle + \frac{1}{2} \langle \Lambda^{2j+2} q_3, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_1, p_1 \right\rangle - \frac{1}{2} \langle \Lambda^{2j+2} q_1, p_3 \rangle \left\langle \Lambda^{2(k-j)} q_1, p_1 \right\rangle \Big].
\end{aligned}
$$

Apparently, $F_{-1} = I$. Furthermore, by the Dirac-Poisson bracket (36) on submanifold M we obtain the following theorem.

Theorem 4 *All Hamiltonian flows (39) and (54) commute on M.*

Proof: Through a lengthy calculation, we have

$$\{H, F_k\}_D = 0,$$

which completes the proof.

Therefore, all Hamiltonian flows (54) are integrable on M. Particularly, the Hamiltonian system (39) is integrable.

Remark 1 *In the calculation procedure and the proof of Theorem 4 and Eq. (54), we used the following facts:*

$$\begin{aligned}
&\{q_1, F\} = \{q_2, F\} = 0,\\
&\{q_3, F\} = \Lambda q_1,\\
&\{p_1, F\} = \{p_2, F\} = 0,\\
&\{p_3, F\} = -\Lambda p_3,\\
&\{F, F_k\} = 0,\\
&\{G, F_k\} = (\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle)(\langle \Lambda^{2k+2} q_1, p_2\rangle + \langle \Lambda^{2k+2} q_2, p_3\rangle),
\end{aligned}$$

and on M all of these equalities $F = G = F_x = F_{xx} = 0$ are valid.

4.3 Parametric solution

Theorem 5 *Let $p(x,t_k), q(x,t_k)$ $(p(x,t_k) = (p_1,p_2,p_3)^T, q(x,t_k) = (q_1,q_2,q_3)^T,\ k = -1,-2,...)$ be the common solution of the two integrable flows (39) and (54), then*

$$m \;=\; 2\frac{\langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1 + p_3\rangle}{\langle \Lambda^2 q_2, p_3\rangle + \langle \Lambda^2 q_1, p_2\rangle}, \tag{55}$$

satisfies the negative order hierarchy (18).

Proof: Noticing the following formulas

$$\begin{aligned}
G_k &= -\frac{1}{6}\left\langle \Lambda^{2k+3} q_1, p_3\right\rangle,\ k = -1,-2,-3,...,\\
\Theta^{-1}\Upsilon G_k &= -\frac{1}{6}\Big(\left\langle \Lambda^{2k+2} q_1, p_2\right\rangle + \left\langle \Lambda^{2k+2} q_2, p_3\right\rangle\Big),\\
\Upsilon &= m\partial + 2\partial m,\quad \Upsilon^* = \partial m + 2m\partial,\quad \Theta = \partial - \partial^3,
\end{aligned}$$

and Eq. (54), we directly compute and find Eq. (55) satisfies $m_{t_k} = 3\Upsilon^*\Theta^{-1}\Upsilon G_k$ which completes the proof.

In particular, we obtain the following theorem.

Theorem 6 *Let $p(x,t), q(x,t)$ $(p(x,t) = (p_1,p_2,p_3)^T, q(x,t) = (q_1,q_2,q_3)^T)$ be the common solution of the two integrable flows (39) and (47), then*

$$\begin{aligned} m &= 2\frac{\langle \Lambda q_2(x,t), p_2(x,t)\rangle - \langle \Lambda q_3(x,t), p_1(x,t)+p_3(x,t)\rangle}{\langle \Lambda^2 q_2(x,t), p_3(x,t)\rangle + \langle \Lambda^2 q_1(x,t), p_2(x,t)\rangle}, \quad (56)\\ u &= \frac{1}{2}\left(\langle q_1(x,t), p_2(x,t)\rangle + \langle q_2(x,t), p_3(x,t)\rangle\right) - \frac{1}{2}H, \quad (57)\end{aligned}$$

satisfy the DP equation:

$$m_t + um_x + 3mu_x = 0. \quad (58)$$

Proof: Let

$$\begin{aligned} A &= \langle \Lambda q_2, p_2\rangle - \langle \Lambda q_3, p_1+p_3\rangle,\\ B &= \left\langle \Lambda^2 q_2, p_3\right\rangle + \left\langle \Lambda^2 q_1, p_2\right\rangle,\\ C &= \left\langle \Lambda^2 q_3, p_3\right\rangle - \left\langle \Lambda^2 q_1, p_1+p_3\right\rangle.\end{aligned}$$

Then through a lengthy calculation we have

$$\begin{aligned} A_t &= uG + umC - 2u_xA = umC - 2u_xA,\\ B_t &= G - uC + u_xB = -uC + u_xB,\\ A_x &= -2G - mC = -mC,\\ B_x &= C.\end{aligned}$$

By the above equalities, we obtain

$$\begin{aligned} & m_t + um_x + 3mu_x\\ &= \frac{2}{B^2}\Big[A_tB - AB_t + u(A_xB - AB_x) + 3u_xAB\Big]\\ &= \frac{2}{B^2}\Big(umBC + uAC + uA_xB - uAB_x\Big)\\ &= \frac{2u}{B^2}\Big(B(mC + A_x) + A(C - B_x)\Big)\\ &= 0.\end{aligned}$$

In the above proof procedures, we have used the following equalities: $F = G = 0,\ F_x = F_{xx} = 0$ on M.

Similarly, we can discuss the parametric solution of the positive order hierarchy (23). That needs us to consider a new kind of constraint and related integrable system, which we deal with in the next section.

5 Parametric solution of the positive order hierarchy (23)

Let us directly consider the following constraint:

$$G_0 = \sum_{j=1}^{N} E_j \nabla \lambda_j, \tag{59}$$

where $E_j \nabla \lambda_j = \lambda_j q_{1j} p_{3j}$ is the functional gradient of λ_j for the spectral problems (8) and (27), and $q_{kj},\ p_{kj}$ $(k = 1, 2, 3)$ are the related eigenfunctions of λ_j. Then Eq. (59) is saying

$$m = \langle \Lambda q_1, p_3 \rangle^{-\frac{3}{2}} \tag{60}$$

which composes a new constraint in the whole space $\mathbb{R}^{6N}$. Under this constraint, the spectral problem (8) and its adjoint problem (27) are able to be cast in a Hamiltonian canonical form in $\mathbb{R}^{6N}$:

$$(H^+): \quad \begin{aligned} q_x &= \{q, H^+\}, \\ p_x &= \{p, H^+\}, \end{aligned} \tag{61}$$

with the Hamiltonian

$$H^+ = \langle q_2, p_1 + p_3 \rangle + \langle q_3, p_2 \rangle + \frac{2}{\sqrt{\langle \Lambda q_1, p_3 \rangle}}. \tag{62}$$

To see the integrability of the system (61), we take into account of the time part $\Psi_t = V_1(m, \lambda)\Psi$ and its adjoint $\Psi_t = -V_1^T(m, \lambda)\Psi$, where $V_1(m, \lambda)$ is defined by Eq. (26). Under the constraint (60), the time part and its adjoint are also nonlinearized as a canonical Hamiltonian system in $\mathbb{R}^{6N}$

$$(F_1): \quad \begin{aligned} q_t &= \{q, F_1\}, \\ p_t &= \{p, F_1\}, \end{aligned} \tag{63}$$

with the Hamiltonian

$$
\begin{aligned}
F_1 \quad = \quad & 6\frac{\langle \Lambda^2 q_1, p_2\rangle + \langle \Lambda^2 q_2, p_3\rangle}{\sqrt{\langle \Lambda q_1, p_3\rangle}} \\
& -\frac{1}{2}(\langle \Lambda q_1, p_1\rangle^2 + 4\langle \Lambda q_2, p_2\rangle^2 + \langle \Lambda q_3, p_3\rangle^2) \\
& +\frac{3}{2}(\langle \Lambda q_1, p_3\rangle^2 + \langle \Lambda q_2, p_3\rangle^2 - \langle \Lambda q_1, p_2\rangle^2) \\
& +3\langle \Lambda q_1, p_3\rangle(\langle \Lambda q_1, p_1\rangle - 2\langle \Lambda q_3, p_1\rangle - \langle \Lambda q_3, p_3\rangle) \\
& +2\langle \Lambda q_1, p_1\rangle\langle \Lambda q_2, p_2\rangle + 2\langle \Lambda q_2, p_2\rangle\langle \Lambda q_3, p_3\rangle - \langle \Lambda q_1, p_1\rangle\langle \Lambda q_3, p_3\rangle \\
& +3(\langle \Lambda q_1, p_2\rangle - \langle \Lambda q_2, p_3\rangle)(\langle \Lambda q_3, p_2\rangle - \langle \Lambda q_2, p_1\rangle). \qquad (64)
\end{aligned}
$$

After a calculation of the Poisson bracket of $\{H^+, F_1\}$, we know that the two canonical Hamiltonian flows commute, i.e.

$$\{H^+, F_1\} = 0.$$

Furthermore, under the constraint (60) the nonlinearized systems of the general time part $\Psi_{t_k} = \sum_{j=0}^{k-1} V(G_j)\lambda^{2(k-j-1)}\Psi$ $(k > 0,\ k \in \mathbb{Z})$ and its adjoint problem produce the following canonical Hamiltonian system in $\mathbb{R}^{6N}$:

$$
(F_k): \quad \begin{array}{l} q_{t_k} = \{q, F_k\}, \\ p_{t_k} = \{p, F_k\}, \end{array} \quad k = 1, 2, 3, ..., \qquad (65)
$$

with the Hamiltonian

$$
\begin{aligned}
F_k \quad = \quad & 6\frac{\langle \Lambda^{2k} q_1, p_2\rangle + \langle \Lambda^{2k} q_2, p_3\rangle}{\sqrt{\langle \Lambda q_1, p_3\rangle}} \\
& +\sum_{j=0}^{k-1}\Big[-\frac{1}{2}\langle \Lambda^{2j+1} q_1, p_1\rangle\left\langle \Lambda^{2(k-j)-1} q_1, p_1\right\rangle - 2\langle \Lambda^{2j+1} q_2, p_2\rangle\left\langle \Lambda^{2(k-j)-1} q_2, p_2\right\rangle - \frac{1}{2}\langle \Lambda^{2j+1} q_3, p_3\rangle\left\langle \Lambda^{2(k-j)-1} q_3, p_3\right\rangle \\
& +2\langle \Lambda^{2j+1} q_2, p_2\rangle\left\langle \Lambda^{2(k-j)-1} q_1, p_1\right\rangle - \langle \Lambda^{2j+1} q_3, p_3\rangle\left\langle \Lambda^{2(k-j)-1} q_1, p_1\right\rangle + 2\langle \Lambda^{2j+1} q_3, p_3\rangle\left\langle \Lambda^{2(k-j)-1} q_2, p_2\right\rangle \\
& -\frac{3}{2}\langle \Lambda^{2j+1} q_1, p_2\rangle\left\langle \Lambda^{2(k-j)-1} q_1, p_2\right\rangle + \frac{3}{2}\langle \Lambda^{2j+1} q_1, p_3\rangle\left\langle \Lambda^{2(k-j)-1} q_1, p_3\right\rangle + \frac{3}{2}\langle \Lambda^{2j+1} q_2, p_3\rangle\left\langle \Lambda^{2(k-j)-1} q_2, p_3\right\rangle \\
& +3\Big(\langle \Lambda^{2j+1} q_1, p_2\rangle - \langle \Lambda^{2j+1} q_2, p_3\rangle\Big)\Big(\left\langle \Lambda^{2(k-j)-1} q_3, p_2\right\rangle - \left\langle \Lambda^{2(k-j)-1} q_2, p_1\right\rangle\Big) \\
& +3\langle \Lambda^{2j+1} q_1, p_3\rangle\Big(\left\langle \Lambda^{2(k-j)-1} q_1, p_1\right\rangle - 2\left\langle \Lambda^{2(k-j)-1} q_3, p_1\right\rangle - \left\langle \Lambda^{2(k-j)-1} q_3, p_3\right\rangle\Big)\Big] \\
& +\sum_{j=1}^{k-1}\Big[-\frac{3}{2}\langle \Lambda^{2j} q_1, p_1\rangle\left\langle \Lambda^{2(k-j)} q_1, p_1\right\rangle - \frac{3}{2}\langle \Lambda^{2j} q_3, p_3\rangle\left\langle \Lambda^{2(k-j)} q_3, p_3\right\rangle \\
& +\frac{3}{2}\langle \Lambda^{2j} q_1, p_2\rangle\left\langle \Lambda^{2(k-j)} q_1, p_2\right\rangle - \frac{3}{2}\langle \Lambda^{2j} q_1, p_3\rangle\left\langle \Lambda^{2(k-j)} q_1, p_3\right\rangle - \frac{3}{2}\langle \Lambda^{2j} q_2, p_3\rangle\left\langle \Lambda^{2(k-j)} q_2, p_3\right\rangle \\
& -3\Big(\langle \Lambda^{2j} q_1, p_2\rangle + \langle \Lambda^{2j} q_2, p_3\rangle\Big)\Big(\left\langle \Lambda^{2(k-j)} q_3, p_2\right\rangle + \left\langle \Lambda^{2(k-j)} q_2, p_1\right\rangle\Big) \\
& +3\langle \Lambda^{2j} q_1, p_3\rangle\left\langle \Lambda^{2(k-j)} q_3, p_3\right\rangle + 3\langle \Lambda^{2j} q_3, p_3\rangle\left\langle \Lambda^{2(k-j)} q_1, p_1\right\rangle - 3\langle \Lambda^{2j} q_1, p_3\rangle\left\langle \Lambda^{2(k-j)} q_1, p_1\right\rangle\Big].
\end{aligned}
$$

Apparently, when $k=1$, F_k is exactly Eq. (64). Furthermore, through a lengthy computation, we obtain

$$\{H^+, F_k\} = 0, \quad k = 1, 2, ..., \tag{66}$$

which represents each Hamiltonian t-flow (F_k) commutes with Hamiltonian x-flow (H^+). Thus, all Hamiltonian canonical systems (F_k) are integrable in $\mathbb{R}^{6N}$. Particularly, the nonlinearized spectral problems (61) is integrable.

Like last section, we also have a similar theorem

Theorem 7 *Let $p(x,t_k), q(x,t_k)$ ($p(x,t_k) = (p_1,p_2,p_3)^T, q(x,t_k) = (q_1,q_2,q_3)^T$, $k = 1,2,3,...$) be the common solution of the two integrable flows (61) and (65), then*

$$m = \frac{1}{\sqrt{\langle \Lambda q_1(x,t_k), p_3(x,t_k)\rangle^3}}, \quad k = 1,2,3,..., \tag{67}$$

satisfy the positive order hierarchy (23).

In particular, we have the following theorem.

Theorem 8 *Let $p(x,t), q(x,t)$ ($p(x,t) = (p_1,p_2,p_3)^T, q(x,t) = (q_1,q_2,q_3)^T$) be the common solution of the two integrable flows (61) and (63), then*

$$m = \frac{1}{\sqrt{\langle \Lambda q_1(x,t), p_3(x,t)\rangle^3}} \tag{68}$$

is a parametric solution of the 5th-order equation:

$$m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}. \tag{69}$$

Proof: A direct check is done through substitution of Eqs. (61) and (63).

6 Peaked stationary solutions

We know that the equation $m_t + um_x + 3mu_x = 0$, $m = u - m_{xx}$ has peakon solution $u = e^{-|x+t|}$. Furthermore, a generalized b-balanced equation $m_t + um_x + bmu_x = 0$, $m = u - m_{xx}, b = constant$ is found to have this kind of solution [12].

Now, we study the traveling wave solution of the equation $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. Set $m^{-\frac{2}{3}} = v$, then this equation becomes

$$-\frac{3}{2}v^{-\frac{2}{3}}v_t = 4v_x - 5v_{xxx} + v_{xxxxx}. \tag{70}$$

Assume this equation has the solution $v = f(\xi),\ \xi = x - ct,\ c = constant$, then we have

$$2cf^{-\frac{1}{2}} = 2f^2 - \frac{5}{2}f'^2 + f'f''' - \frac{1}{2}f''^2. \tag{71}$$

The right hand side of this equation is quadratically homogeneous and the left hand side not. Therefore, we set $f = e^{a\xi},\ a = constant$ and substitute it into Eq. (71), and obtain

$$\begin{aligned} c &= 0, &(72)\\ a^4 - 5a^2 + 4 &= 0, &(73) \end{aligned}$$

which implies

$$a = \pm 1, \quad a = \pm 2. \tag{74}$$

So, the 5th-order equation (70) has the following stationary solutions:

$$1,\ e^{-x},\ e^{x},\ e^{-2x},\ e^{2x}, \tag{75}$$

which exactly composes the basis of the solution space of the stationary equation $4v_x - 5v_{xxx} + v_{xxxxx} = 0$. Therefore, the 5th order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$ possesses the stationary solutions

$$\begin{aligned} m(x) &= (c_0 + c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{2x})^{-\frac{3}{2}}, \qquad (76)\\ &\quad \forall\ c_k \in \mathbb{R},\ k = 0, 1, 2, 3, 4. \end{aligned}$$

Apparently, $e^{-\frac{3}{2}x}$, $e^{\frac{3}{2}x}$, e^{-3x}, e^{3x} satisfy the 5th order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. Therefore, $f_1^{\wedge}(x) = e^{-\frac{3}{2}|x|}$, $f_1^{\vee}(x) = e^{\frac{3}{2}|x|}$, $f_2^{\wedge}(x) = e^{-3|x|}$, and $f_2^{\vee}(x) = e^{3|x|}$ are the four peaked stationary solutions (see Figure 1) of the 5th-order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$.

Comparison with the parametric solution (68)

All the stationary solutions (76) may be included in the parametric solution (68). For example, the function $m(x) = e^{-\frac{3}{2}x}$ is cast in Eq. (68) when we choose dynamical variables $q_1,\ p_3$ such that the following constraint

$$\langle \Lambda q_1(x), p_3(x) \rangle = e^{x} \tag{77}$$

holds, where $q_1,\ p_3$ are the solutions of the integrable Hamiltonian system (61). But the four peaked stationary solutions can not be included in the parametric solution (68) because $\langle \Lambda q_1(x), p_3(x) \rangle$ is smooth every where.

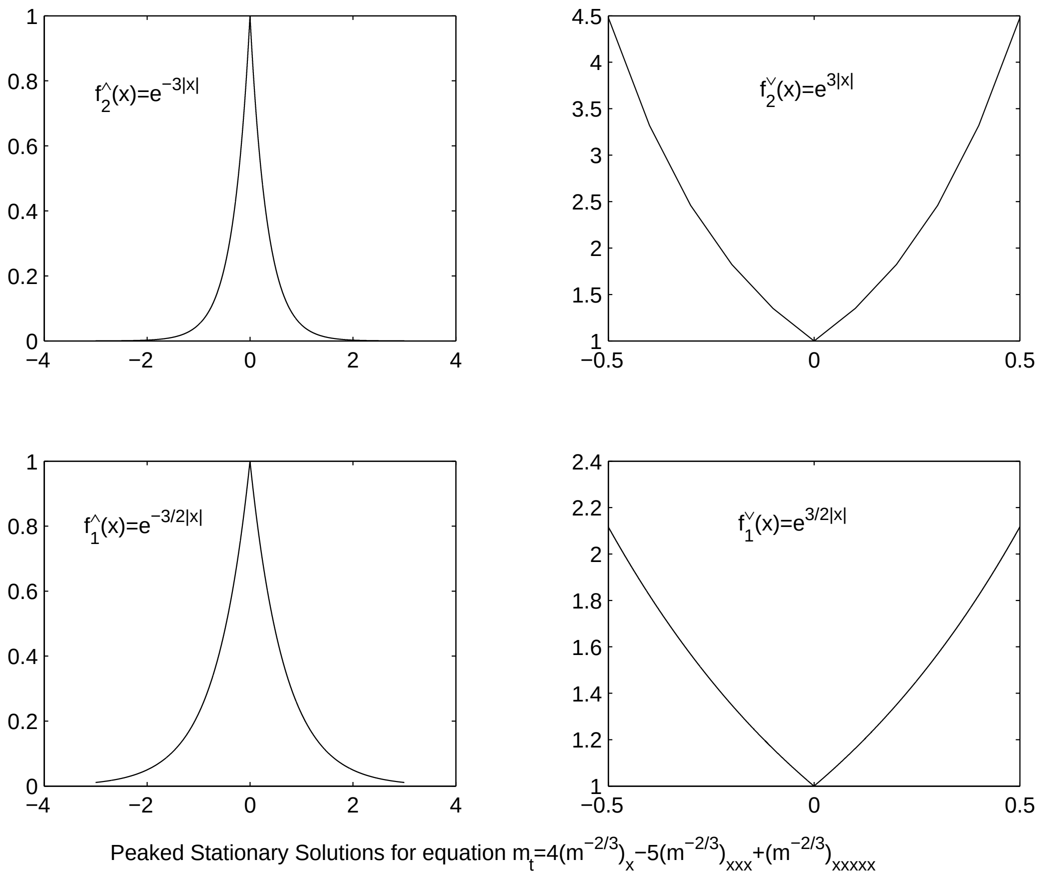

Figure 1: Four peaked stationary solutions

Comparison with the cusp solution [22]

Wadati, Ichikawa and Shimizu [22] proposed the cusp soliton (cuspon) solution for the deformed Harry-Dym equation $r_t + (1-r)^3 r_{xxx} = 0$. This equation is actually equivalent to the Harry-Dym equation $q_t - 2(q^{-1/2})_{xxx} = 0$ by the transformation $q^{-1/2} = 1 - r$. By the inverse scattering method [2], they obtained the following traveling wave solution

$$r = \cosh^{-2}\xi,\ \xi = ax - 4a^3 t + \tanh\xi + A, \tag{78}$$

where $a \neq 0, a,\ A = \textit{constants}$. They called this solution the cusp soliton. It seems that the present 5th-order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$ looks like a higher-order Harry-Dym equation and should have the cusp-kind solution (78). But unfortunately, this is not the case for our 5th-order PDE $m_t = 4(m^{-\frac{2}{3}})_x - 5(m^{-\frac{2}{3}})_{xxx} + (m^{-\frac{2}{3}})_{xxxxx}$. However, our recent study [10] reveals that the following

5th-order PDE

$$(15r^2 + 18r + 2)r_t + (1 - r)^6 r_{xxxxx} = 0 \tag{79}$$

possesses the cuspon $r = \cosh^{-2}\xi$ with $\xi = ax - 8a^5 t + \tanh\xi + A$, $\forall a \neq 0, A \in \mathbb{R}$.

Acknowledgments

We would like to express our sincere thanks to Prof. Konopelchenko for his suggestion and Prof. Magri for his fruitful discussion during their visit at Los Alamos National Laboratory.

This work was supported by the U.S. Department of Energy under contracts W-7405-ENG-36 and the Applied Mathematical Sciences Program KC-07-01-01; and also the Special Grant of National Excellent Doctorial Dissertation of PR China.

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